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On the Geometric and Algebraic Foundation of the Spinor Formalism and the Application to Relativistic Field Equations

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Abstract

The Hilbert calculus of segments plays an important role in the axiomatic foundation of the Euclidean geometry, as the relationship to some fundamental agebraic structures can be made more apparent. An extension of the Hilbert calculus to the field of the quaternions $U2$ or biquaternions $U4$ leads to some new aspects on the spinor formalism. By that, a geometrical interpretation of the Dirac equation is obtained. Including the torsion of the Minkowski space (Cartan geometry), the affine connection of the spinor space $U4$ also can be interpreted with the help of a generalized Hilbert calculus. These considerations lead to a simple geometrical access to the nonlinear spinor theory, proposed by Ivanenko, Heisenberg, Dürr, etc.

1. *Introduction*

Nonlinear relativistic field equations have become an important tool for the description of interaction phenomena in elementary particle physics. The first nonlinear generalization of the Dirac equation ($\hbar = c = 1$)

$$i\gamma^{\nu}(\Psi_{|\nu} \pm eA_{\nu}\Psi) = m\Psi \tag{1.1}$$

where γ^{ν} satisfies the algebra

$$\begin{gathered}\gamma_{\nu}\gamma_{\lambda} + \gamma_{\lambda}\gamma_{\nu} = 2\tilde{g}_{\nu\lambda}\\ \tilde{g}_{\nu\lambda} = \epsilon_{\nu}\delta_{\nu\lambda}, \qquad \epsilon_{\nu} = (1, 1, 1, -1)\end{gathered} \tag{1.2}$$

has been investigated by Ivanenko (1938) and Ivanenko and Brodski (1957) by adding the term of the form $\sim\Psi^3$.

In recent time, similar attempts in this direction are made by the so-called constructive quantum field theory (Glimm et al., 1973), where nonlinear generalizations of the Klein-Gordon equation

$$\Box\Psi + m^2\Psi = \lambda\Psi^3$$

are taken into account, but our interest lies in the investigation of the spinor formalism. A nonlinear spinor equation

$$\gamma^\nu\Psi_{|\nu} \pm l^2\gamma^\lambda\gamma(\bar{\Psi}\gamma_\lambda\gamma\Psi)\Psi = 0 \tag{1.3}$$

for the description of the interaction processes between the elementary particles has been proposed by Heisenberg (Dürr, 1976). In equation (1.3) γ^ν means the direct product of spin and isospin. (However, the question of the interpretation of metrical bispin tensors has no importance in this paper.) Equation (1.3) yields the following conservation laws:

$$(\bar{\Psi}\gamma^\nu\Psi)_{|\nu} = 0 \tag{1.4}$$

and

$$(\bar{\Psi}\gamma^\nu\gamma\Psi)_{|\nu} = 0 \tag{1.5}$$

Equation (1.4) is a consequence of the Hermitean Dirac operator and represents the conservation of the electric charge. Equation (1.5) is sometimes connected with the conservation of the baryonic charge. The two conservation laws are obtained by a general gauge transformation

$$\tilde{\Psi} = \exp(-i\chi)A\Psi$$
$$A = \exp(i\eta\gamma) \tag{1.6}$$

where χ is a real gauge function and η a real number. The transformation, given by the matrix A, is known as a Touschek transformation. It has been shown by several authors (Rodichev, 1961; Braunss, 1965; Schmutzer, 1968; Hehl et al., 1974; Ulmer, 1975) that the nonlinear term of equation (1.3) represents the affine connection in the spinor formalism of a Minkowski space X_4 with torsion (Minkowski-Cartan geometry):

$$\Gamma_\nu{}^c = \gamma(\bar{\Psi}\gamma_\nu\gamma\Psi) \tag{1.7}$$

$\Gamma_\nu{}^c$ is induced by a parallel displacement of the vector space U_4 of bispinors Ψ, embedded in a Minkowski-Cartan geometry, and in this case the covariant derivation is written in the form

$$\Psi_{\|\nu} = \Psi_{|\nu} \pm \Gamma_\nu{}^c\Psi \tag{1.8}$$

A possible way to obtain equation (1.3) is the variation of the corresponding Lagrangian:

$$L = \tfrac{1}{2}[\bar{\Psi}\gamma_\mu(\bar{\Psi}_{|\mu} \pm \Gamma_\mu{}^c\Psi) - (\bar{\Psi}_{|\mu} \pm \bar{\Psi}_\mu\bar{\Gamma}_\mu{}^c)\gamma_\mu\Psi] \tag{1.9}$$

The aim of the following considerations is the investigation of the geometric, algebraic structure of the parallel displacement $\Gamma_\mu{}^c$. By that, the geometric background of the equations (1.1) and (1.3) can be made more apparent.

2. *Basic Considerations*

A linear vector space $V_L(\mathbb{R})$ over the field of real numbers $\mathbb{R}$ can be equipped with a topology by various methods, e.g., with the aid of a norm

$$g: V_L \times V_L \to \mathbb{R}_+ \tag{2.1}$$

where g represents a mapping of $V_L(\mathbb{R}) \to \mathbb{R}_+$ and $\mathbb{R}_+$ is the set of positive real numbers. In physical problems, the Euclidean norm and related generalizations (e.g., Riemannian geometry, Hilbert space) are most frequently used. Therefore we consider some algebraic relations of the Euclidean geometry in the axiomatic formulation given by Hilbert (see Beth, 1965). For this purpose, it should be pointed out that the axiomatic frame of the Euclidean geometry is founded by the following special relations: The axioms of connection, order, congruence, parallelism and continuity. According to Hilbert an area measure is obtained by a suitable multiplication of straight lines (the Hilbert calculus of segments), where the continuity axiom does not have to be used. The Hilbert calculus of segments makes apparent the relationship of the above axioms with some fundamental algebraic structures and the integral calculus. In Figure 1, we

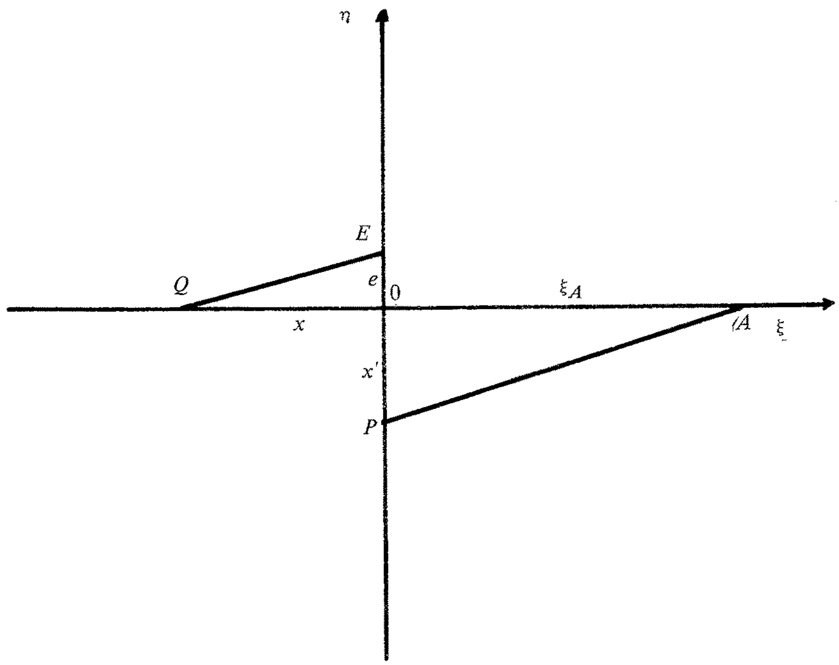

Figure 1.

regard the points P, Q, E, A, O in a right-angular coordinate system (ξ, η). $\overline{QO} = x$ and $\overline{PO} = x'$ are arbitrary straight lines, and $\overline{OE} = e$ is the unity straight line. A parallel displacement of $\overline{QE}$ at the point $P(\overline{QE} \parallel \overline{PA})$ defines the straight line $\xi_A = \overline{OA}$. The axioms of the Euclidean geometry yield the ratio:

$$x : e = \xi_A : x' \tag{2.2}$$

and equation (2.2) is equivalent to

$$xx' = e\xi_A = \xi_A e = \xi_A \tag{2.2a}$$

$\xi_A = xx'$ represents a measure of the area of the segment with the sides x and x'. In Figure 2 the reversal construction of the segment $\eta_A = x'x$ can be verified.

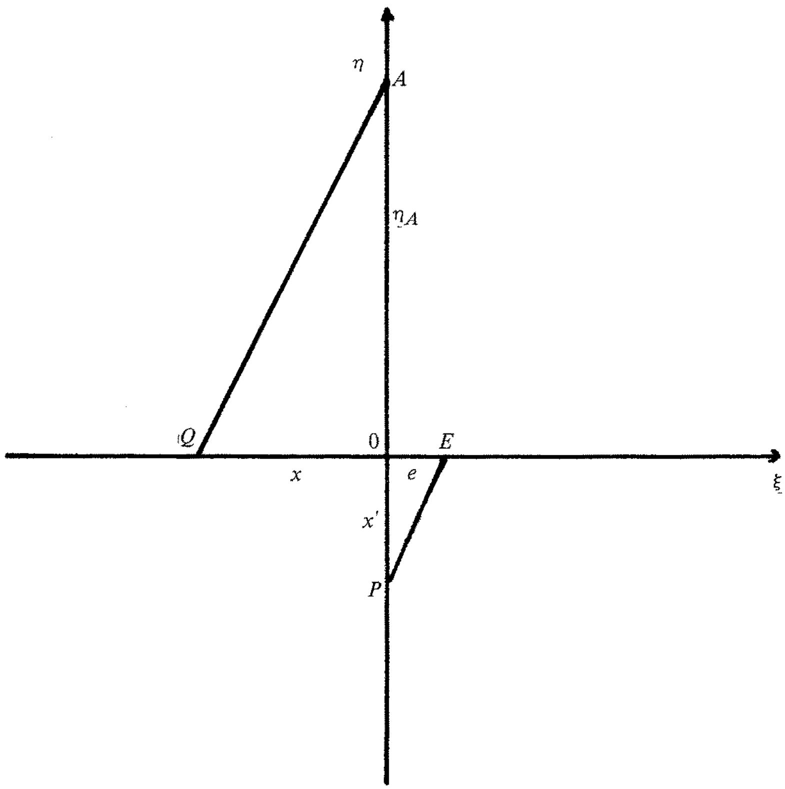

Figure 2.

The unity straight line $e = \overline{OE}$ must be chosen on the ξ axis, and the parallel displacement of $\overline{PE}$ at the point $Q(\overline{QA} \parallel \overline{PE})$ yields the following ratio:

$$x' : e = \eta_A : x \tag{2.3}$$

which is equivalent to

$$x'x = e\eta_A = \eta_A e = \eta_A \tag{2.3a}$$

It is necessary to point out that ξ_A and η_A represent straight lines again, and their amounts are proportional to the corresponding segments xx' and $x'x$. Furthermore, it is very important from the axiomatic point of view that the segments xx' and $x'x$ are constructed in a quite different way, and the relationship to certain algebraic structures is evident: The satisfaction of the commutativity law requires $\xi_A - \eta_A = 0$. The associativity law yields a scaling transformation $x = xs$ or $x' = sx'$: $(xs)x' = x(sx') = xsx'$. (s must commute with x' and x, as, e.g., $x's = sx'$ must hold.) The distributivity law is obtained by a translation $x \Rightarrow x + \lambda$:

$$(x + \lambda)x' = xx' + \lambda x'$$

This example can readily be generalized to the projective and affine geometry, respectively, to the affine connection in the Riemannian geometry. A further application is the mean value theorem of the integral calculus. Let x' be a real, continuous function over $\mathbb{R}: x' = f(x)$. An interval $I \subset \mathbb{R}$ is given by $I = (0 \leqq \xi \leqq x)$. There exists a real number $\tilde{x} \in I$, for which the relation

$$xf(\tilde{x}) = \int_{0}^{x} f(\xi)\, d\xi = \eta_A \tag{2.4}$$

is valid.

In the case of the Riemannian geometry the Hilbert calculus of straight lines can only be used for infinitesimal segments, induced by a translation $x \Rightarrow x + dx$. Finite parallel displacements must be interpreted in the sense of the mean value theorem. The concept of parallel displacement or parallel transfer of a certain object χ (spinor, tensor) is induced by the definition of the covariant derivation, since the partial derivation $\chi_{|\nu}$ is not covariant:

$$\begin{aligned} D\chi &= \chi_{\|\nu}\, dx^{\nu} \\ D\chi &= \chi_{|\nu}\, dx^{\nu} + \Gamma_{\nu}\chi\, dx^{\nu} \end{aligned} \tag{2.5}$$

If the quantity χ remains invariant under a parallel transfer, then the condition $D\chi = 0$ must hold (generalized parallelism). In general, parallel transfer is a path-dependent concept, and the difference $D\chi = \chi(x + dx) - \chi(x)$ does not vanish. The main interest of the following considerations lies in the geometric and algebraic structure of the affine connection, if χ belongs to the vector space $U2$ or $U4$. The Hilbert calculus of segments is generalized to other structures, as the straight lines can also be interpresented as elements of the quaternion field $U2$. In the case of the Euclidean geometry we have observed that from the multiplication of any two straight lines there results a new straight line, and therefore the application to other structures is possible. The quaternion field $U2$ is the only noncommutative field over $\mathbb{R}$ with a finite number of basis elements. The identification of the straight lines with the elements of the quaternion field $U2$ requires the abandonment of the commutativity law, and the multiplication of these "straight lines" may be represented in the following manner:

$$U2(\mathbb{R}) \times U2(\mathbb{R}) \rightarrow U2(\mathbb{R}) \tag{2.6}$$

There exist four basis elements (σ, e), satisfying the commutation rule

$$\begin{aligned} \sigma_j \sigma_j &= e \qquad (j = 1, 3) \\ \sigma_j \sigma_k - \sigma_k \sigma_j &= 2i\sigma_l \qquad (j, k, l \text{ cyclic}) \end{aligned} \tag{2.7}$$

According to a fundamental theorem of Wedderburn each field with a *finite* number of elements *must* be commutative with respect to multiplicative operation. This fact is very important and must be taken into account, respectively, if the mapping of a linear vector space (e.g., X_4) on a *noncommutative* field is considered. In this case, the multiplication of a vector with the corresponding scalar does not have to be commutative, if the field F', basing on the vector space $V_L(F)$, is not finite. For the following, the Kronecker product of any linear space $V_L(\mathbb{R})$ with $U2(\mathbb{R})$ or the biquaternion field $U4(\mathbb{R})$ is taken into consideration:

$$\begin{aligned} V_L \otimes U2 &\to U2 \\ V_L \otimes U4 &\to U4 \end{aligned} \tag{2.8}$$

For each $x^\nu \in V_L$ relation (2.8) yields the quantity

$$x = \sigma_\mu \otimes x^\mu = \sum_\mu \sigma_\mu x^\mu \tag{2.9}$$

or

$$x = \gamma_\mu \otimes x^\mu = \sum_\mu \gamma_\mu x^\mu \tag{2.9a}$$

As already stated, the reversal construction represents an important example for the fact that the multiplication of any element $x' \in V_L'$ with x, defined by the relation (2.9) or (2.9a) does not have to be commutative:

$$V_L' \times (V_L \otimes U4) - (V_L \otimes U4) \times V_L' \to U4 \tag{2.10}$$

With the help of equation (2.9a) we can form the commutation rule, satisfying the algebra of relation (2.10): $(x'^\nu \in V_L', x^\nu \in V_L)$

$$\gamma_\mu x^\mu x'^\nu - x'^\nu \gamma_\mu x^\mu = \gamma^\nu \tag{2.11}$$

In the following, we shall show that the Dirac equation (1.1) and the nonlinear spinor equation (1.3) can be considered as realizations of the algebraic structure (2.11).

3. *Applications to U4 with Minkowski-Cartan Geometry*

In this section, we denote a vector space with Minkowskian geometry by X_4. If torsion of X_4 (Cartan geometry) is taken into consideration, the four-dimensional manifold with Minkowski-Cartan geometry is denoted by ${X_4}^c$. In the case of the Dirac equation (1.1) it is sufficient to consider only the biquaternion field $U4$ on X_4. Let $x^\nu \in X_4$ and p'^μ $(\mu = 1, 4)$ be the momentum of a particle in the rest frame; then the application of the algebraic structure (2.11) yields the commutator

$$x^\nu \gamma_\mu p'^\mu - \gamma_\mu p'^\mu x^\nu = K\gamma^\nu \tag{3.1}$$

K is a constant and must be chosen to agree with physical dimensions (e.g., Planck's constant). In the case of the Dirac equation we may put $K = i$, and, since $P^{\mu\prime} = (0, 0, 0, m)$, we obtain from (3.1)

$$mx^\nu - x^\nu m = -i\gamma^\nu$$

or (3.1a)

$$mx^\nu - x^\nu m = i\gamma^\nu$$

Replacing m by the differential operator, the Dirac equation

$$i\gamma^\nu \Psi_{|\nu} = \pm m\Psi$$

for a free particle is obtained. The vector potential A_ν can also be introduced by a gauge transformation of the commutator (3.1).

The more general case is given by a torsion of the Minkowski space $X_4{}^c$. The parallel transfer (affine connection in the spinor formalism) is not a commutative structure in the sense of the Hilbert calculus of straight lines. The application of the relation (2.11) leads to the commutator

$$x^\nu \gamma_\mu \Gamma^{c\mu} - \gamma_\mu \Gamma^{c\mu} x^\nu = l^{-2}\gamma^\nu \tag{3.2}$$

Owing to well-known arguments (see Rodichev, 1961; Braunss, 1965; Schmutzer,, 1968; Hehl et al., 1974; and Ulmer, 1975) $\Gamma_\mu{}^c$ must be an axial vector ($\bar{\Gamma}_\mu{}^c = -\Gamma_\mu{}^c$) and is defined by equation (1.7). Making use of the well-known representation of algebraic commutators by differential operators, the nonlinear spinor equation (1.3) is obtained:

$$l^{-2}\gamma^\nu \phi_{|\nu} = -\gamma_\mu \Gamma^{c\mu}\phi \tag{3.3}$$

Assuming that ϕ and Ψ differ by a transformation $\phi = A\Psi$, we find that A must agree with the already mentioned Touschek transformation, as the affine connection $\Gamma_\mu{}^c$ remains invariant under this transformation. The differentiation of the commutator (3.2) leads to

$$\delta^\nu_{\nu'}\gamma_\mu \Gamma^{c\mu} + x^\nu \gamma_\mu \Gamma^{c\mu}_{|\nu'} - \gamma_\mu \Gamma^{c\mu}\delta^\nu_{\nu'} - \gamma_\mu \Gamma^{c\mu}_{|\nu'} x^\nu \equiv 0 \tag{3.4}$$

As the following relations

$$x^\nu \gamma_\mu \Gamma^{c\mu}_{|\nu'} - \gamma_\mu \Gamma^{c\mu}_{|\nu'} x^\nu \equiv 0$$

must be valid for each $x^\nu \in X_4$, we find that

$$\Gamma^{c\mu}_{|\mu} = 0 \Rightarrow (\bar{\Psi}\gamma^\mu \gamma \Psi)_{|\mu} = 0 \tag{3.5}$$

is valid for equation (1.3), as already stated with the help of other means. The above relation (3.5) directly follows from the commutator (3.2), which represents the algebraic structure of the nonlinear spinor theory. We should finally note that the axiomatic foundation of the geometry makes the geometric, algebraic background of relativistic quantum theory more distinct. As the parallel transfer of physical quantities (tensors, spinors) also plays a significant role in general relativity, a unified aspect of modern field problems can be built up by algebraic structures.

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